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Solutio problematis, quo duo quaeruntur numeri, quorum productum tam summa quam differentia eorum sive auctum sive minutum fiat quadratum

Leonhard Euler

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S O L V T I O P R O B L E M A T I S,

QVO DVO QVAERVNTVR NVMERI, QVORVM PRODVCTVM
TAM SVMMA, QVAM DIFFERENTIA EORVM, SIVE AVCTVM
SIVE MINVTVM FIAT QVADRATVM.

Auctore

L. E V L E R O.

1. Problema hoc mihi ante complures annos Berolini a Centurione quodam Prussico erat propositum, quod se, Lipsiae ab amico accepisse aiebat; neque vero se neque istum amicum solutionem villo modo inuenire potuisse. Quaerebat igitur ex me vtrum hoc Problema possibile iudicarem nec ne? Statim quidem hoc problema mihi ob elegantiam mirifice placebat et quam facile summam solutionis difficultatem perspexissem, id omnino dignum iudicavi in quo vires meas exercerem. Tandem vero post plura tentamina solutionem sum adeptus, quae ita se habebat: Positis duobus numeris quaesitis A et B, inueni

$$A = \frac{13 \cdot 29^2}{8 \cdot 9^2} = \frac{10933}{648} \text{ et } B = \frac{5 \cdot 29^2}{32 \cdot 11^2} = \frac{4205}{3872}.$$

2. Via autem qua ad hanc solutionem perveni, ita erat comparata, vt nullo modo mihi liceret, alias solutiones inde eruere; etiamsi nullus dubitandi locus relinqueretur,

retur, quin hoc problema innumerabiles admitteret solutiones. Nuper autem cum in hoc idem argumentum incidissem, casu prorsus fortuito methodus mihi se obtulit, infinitas solutiones huius Problematis eliciendi. Quod quum casui prorsus singulari sit acceptum referendum, quaestio haec omnino digna mihi est visa, quam accuratius perscrutarer. Quare primo quidem solutionem generalem proponam, deinde vero artificium illud, quod mihi infinitas solutiones suppeditauit, vberius euoluam.

Solutio Problematis generalis.

3. Si litterae A et B denotent ambos numeros quae-
sitos, necesse est, vt sequentes quatuor formulae quadrata
efficiantur:

$$\begin{aligned} \text{I. } AB + A + B &= \square; & \text{II. } AB + A - B &= \square; \\ \text{III. } AB - A + B &= \square; & \text{IV. } AB - A - B &= \square. \end{aligned}$$

Quum autem statim pateat, hos numeros integros esse non
posse, ob rationes mox perspiciendas, eos ita expressos as-
sumo, vt sit $A = \frac{z}{x}$ et $B = \frac{y}{x}$, ita vt quatuor sequentes
formulae ad quadrata reducendae habeantur:

$$\begin{aligned} \text{I. } \frac{z}{xy}(z + y + x) &= \square; & \text{II. } \frac{z}{xy}(z + y - x) &= \square; \\ \text{III. } \frac{z}{xy}(z - y + x) &= \square; & \text{IV. } \frac{z}{xy}(z - y - x) &= \square. \end{aligned}$$

4. Quod si ergo factor communis fuerit quadratum,
quatuor sequentes formulas quadrata effici oportet, quas
qui-

quidem per ambiguitatem signorum ita duabus formulis comprehendere licet:

$$I. \text{ et } II. \ z + y \pm x = \square; \text{ III. et IV. } z - y \pm x = \square.$$

Quare quum in genere sit $aa + bb \pm 2ab = \square$ similique modo $cc + dd \pm 2cd = \square$; statuamus vt sequitur:

$$z + y = aa + bb; \quad x = 2ab$$

$$z - y = cc + dd; \quad x = 2cd.$$

Vt autem fiat $2ab = 2cd$, statuatur vtrumque $= 2pqrs = x$, sumaturque $a = pq$; $b = rs$; $c = pr$; et $d = qs$ eritque

$$z + y = aa + bb = ppqq + rrss \text{ et}$$

$$z - y = cc + dd = pprp + qqss \text{ vnde colligitur}$$

$$z = \frac{(pp + ss)(qq + rr)}{2} \text{ et } y = \frac{(pp - ss)(qq - rr)}{2}$$

tum vero erit

$$I. \ z + y + x = (a + b)^2 = (pq + rs)^2$$

$$II. \ z + y - x = (a - b)^2 = (pq - rs)^2$$

$$III. \ z - y + x = (c + d)^2 = (pr + qs)^2$$

$$IV. \ z - y - x = (c - d)^2 = (pr - qs)^2.$$

5. Superest igitur, vt etiam factor communis $\frac{z}{xy}$ quadratum reddatur, qui euolutus praebebat hanc formulam:

$$\frac{z}{xy} = \frac{(pp + ss)(qq + rr)}{2pqrs(pp - ss)(qq - rr)}$$

at vero in hoc efficiendo summa consistit difficultas; quodsi enim numerator in denominatorem ducatur, vt haec formula quadratum fieri debeat:

$$2pqrs$$

$2pqrs(pp - ss)(qq - rr)(pp + ss)(qq + rr) = \square$
 singulae litterae ad quinque dimensiones assurgunt, cuius-
 modi quaestiones in Analysisi Diophantea adhuc non sunt
 tractari solitae; ceterum iam olim post plura tentamina re-
 peri huic conditioni satisfieri, sumendo $p=13, s=11, q=16,$
 et $r=11$, vti periculum facienti mox patebit.

6. Quodsi autem quocunque modo huiusmodi valores
 idonei pro literis $p; q; r; s$ fuerint inuenti, solutio pro-
 blematis inde ita adstruitur:

Posita formula $\frac{(pp+ss)(qq+rr)}{2pqrs(pp-ss)(qq-rr)} = \frac{M^2}{N^2}$, primo ambo
 numeri quaesiti, ita erunt expressi

$$A = \frac{(pp+ss)(qq+rr)}{4pqrs} \text{ et } B = \frac{(pp+ss)(qq+rr)}{(pp-ss)(qq-rr)}$$

tum vero conditionibus problematis ita satisfiet vt sit,

- I. $\sqrt{(AB + A + B)} = \frac{M}{N} (pq + rs)$
- II. $\sqrt{(AB + A - B)} = \frac{M}{N} (pq - rs)$
- III. $\sqrt{(AB - A + B)} = \frac{M}{N} (pr + qs)$
- IV. $\sqrt{(AB - A - B)} = \frac{M}{N} (pr - qs).$

Singularis Euolutio nostrae formulae, quae ad
 quadratum est reuocanda.

7. Quum omnis opera in hac formula reducenda fru-
 stra consumatur, quamdiu in ea tot diversae quantitates
 occurrunt, earumque singulae ad tot dimensiones assur-
 gunt, ante omnia elaborandum est, vt diuersis factoribus
 de-

denominatoris communes diuisores concilientur; hunc in finem vsus sum sequentibus positionibus: $p + s = \alpha\beta$; $p - s = \varepsilon\zeta$; $q + r = \alpha\gamma$; et $q - r = \varepsilon\eta$, ita vt fiat $p = \frac{\alpha\beta + \varepsilon\zeta}{2}$; $s = \frac{\alpha\beta - \varepsilon\zeta}{2}$; $q = \frac{\alpha\gamma + \varepsilon\eta}{2}$ et $r = \frac{\alpha\gamma - \varepsilon\eta}{2}$; tum vero nostra conditio principalis postulat, vt sit:

$$\frac{(pp+ss)(qq+rr)}{2pqrs \cdot \beta\gamma\zeta\eta \cdot \alpha^2\varepsilon^2} = \frac{M^2}{N^2} \text{ siue}$$

$$\frac{(pp+ss)(qq+rr)}{2pqrs \cdot \beta\gamma\zeta\eta} = \frac{M^2}{N^2} \cdot \alpha^2\varepsilon^2.$$

8. Secundo constituatur ratio inter litteras r et s , quae sit vt $f:g$, critque $f:g::\alpha\gamma - \varepsilon\eta:\alpha\beta - \varepsilon\zeta$ siue $g(\alpha\gamma - \varepsilon\eta) = f(\alpha\beta - \varepsilon\zeta)$, vnde colligitur $\alpha(f\beta - g\gamma) = \varepsilon(f\zeta - g\eta)$, quocirca ponamus: $\alpha = f\zeta - g\eta$; $\varepsilon = f\beta - g\gamma$; tum vero habebitur

$$p = \frac{2f\beta\zeta - g\beta\eta - g\gamma\zeta}{2}; \quad q = \frac{f\beta\eta + f\zeta\gamma - 2g\gamma\eta}{2};$$

$$r = \frac{f(\gamma\zeta - \beta\eta)}{2} \text{ et } s = \frac{g(\gamma\zeta - \beta\eta)}{2}.$$

9. Vt adhuc plures factores in denominatore communes reddamus; faciamus insuper $q = h\beta\zeta$ vnde haec aequatio emergit: $2h\beta\zeta = f\beta\eta + f\zeta\gamma - 2g\gamma\eta$ siue $\beta(2h\zeta - f\eta) = \gamma(f\zeta - 2g\eta)$ quam ob rem ponamus $\beta = f\zeta - 2g\eta$ et $\gamma = 2h\zeta - f\eta$. Ex his autem valoribus porro colligimus: $\alpha = f\zeta - g\eta$; $\varepsilon = (ff - 2gh)\zeta - fg\eta$; $p + s = (f\zeta - g\eta)(f\zeta - 2g\eta) = ff\zeta\zeta - 3fg\zeta\eta + 2gg\eta\eta$; $p - s = \zeta((ff - 2gh)\zeta - fg\eta) = (ff - 2gh)\zeta\zeta - fg\zeta\eta$; $q + r = (f\zeta - g\eta)(2h\zeta - f\eta) = 2fh\zeta\zeta - (ff + 2hg)\zeta\eta + fg\eta\eta$; $q - r = \eta((ff - 2gh)\zeta - fg\eta) = (ff - 2hg)\zeta\eta - fg\eta\eta$

hincque porro :

$$p = (ff - gh) \zeta \zeta - 2fg\zeta\eta + gg\eta\eta$$

$$s = gh\zeta\zeta - fg\zeta\eta + gg\eta\eta = g(h\zeta\zeta - f\zeta\eta + g\eta\eta)$$

$$q = fh\zeta\zeta - 2gh\zeta\eta = h\zeta(f\zeta - 2g\eta)$$

$$r = fh\zeta\zeta - ff\zeta\eta + fg\eta\eta = f(h\zeta\zeta - f\zeta\eta + g\eta\eta)$$

10. Denique hos valores ita determinemus, vt numerus p diuisor euadat formulae $qq + rr$, iam vero inuenitur:

$$qq + rr = ffgg\eta^4 - 2f^3g\eta^3\zeta + (f^4 + 2ffgh + 4ggghh)\eta\eta\zeta\zeta \\ - 2fh(ff + 2gh)\eta\zeta^2 + 2ffh\zeta^4$$

quare quum sit $p = gg\eta\eta - 2fg\eta\zeta + (ff - gh)\zeta\zeta$, vt p fiat factor illius formulae, statuatur alter factor $ff\eta + t\zeta + u\zeta\zeta$, eritque productum :

$$ffg\eta^4 - 2f^3g\eta^3\zeta + (f^4 - ffg\eta)\eta\eta\zeta\zeta + t(ff - gh)\eta\zeta^2 + u(ff - gh)\zeta^4 \\ + tgg\eta^3 - 2tfg\eta^2\zeta - 2ufg\eta\zeta \\ + ugg\eta^2$$

vbū primū terminū iam congruunt, secundū vero dant $t = 0$,
 tertiū $3ffgh + 4ggghh = ugg$, vnde $u = \frac{3ffb}{g} + 4hh$; quartū
 porro praebent $u = \frac{b(ff + 2gb)}{g}$; quintū vero tandem dant:
 $u = \frac{2ffbh}{ff - gb}$. Necesse igitur est, vt hī tres valores ipsius
 u inter se congruant, primus vero cum secundo collatus
 dat $3ffh + 4ghh = hff + 2ghh$, seu $2ffh + 2ghh = 0$,
 ideoque $ff + gh = 0$; at secundus tertio aequatus dat
 $f^4 - ffg\eta - 2ggghh = 0$, siue $(ff + gh)(ff - 2gh) = 0$,
 vtrin-

utrinque ergo conditioni satisfiat uno eodemque valore

$$h = -\frac{ff}{g}$$

11. Quoniam igitur inuenimus $h = -\frac{ff}{g}$ reliqui valores sequenti modo exprimentur:

$$p = 2ff\zeta\zeta - 2fg\zeta\eta + gg\eta\eta$$

$$q = -\frac{ff}{g} \cdot \zeta (f\zeta - 2g\eta) = 2ff\zeta\eta - \frac{f^3}{g} \cdot \zeta\zeta$$

$$r = -\frac{f^3}{g} \cdot \zeta\zeta - ff\zeta\eta + fg\eta\eta$$

$$s = -ff\zeta\zeta - fg\zeta\eta + gg\eta\eta,$$

vbi notatu dignum euenit, vt in valoribus p et s producta $f\zeta$ et $g\eta$, tamquam simplices quantitates occurrant, quod quidem in litteris q et r non accidit. Verum quia totum negotium, tantum in ratione q ad r versatur, hi ambo valores multiplicentur per $-\frac{g}{f}$, vt sit $q = ff\zeta\zeta - 2fg\zeta\eta$ et $r = ff\zeta\zeta + fg\zeta\eta - gg\eta\eta$; hanc ob rem vt formulas nostras in compendium redigamus atque adeo ad duas quantitates reuocemus, statuamus $f\zeta = m$ et $g\eta = n$, quo facto nostrae quatuor literae ita se habebunt:

$$p = 2mm - 2mn + nn; q = mm - 2mn = m(m - 2n);$$

$$s = -mm - mn + nn; r = mm + mn - nn.$$

12. Quoniam vero res eodem redit siue quaequam littera positue, siue negatiue accipiatur, ponamus

$$p = 2mm - 2mn + nn; q = mm - 2mn = m(m - 2n)$$

$$s = r = mm + mn - nn; \text{ vnde fit}$$

$$p + s = 3mm - mn = m(3m - n)$$

E 2

p —

$$p - s = mm - 3mn + 2nn = (m - n)(m - 2n)$$

$$q + r = 2mm - mn - nn = (m - n)(2m + n)$$

$$q - r = -3mn + nn = -n(3m - n)$$

Hic signum negationis in valore $q - r$, nihil plane turbat, tantum enim opus est litteras q et r inter se permu-
tari, ita vt sit

$$p = 2mm - 2mn + nn; q = mm + mn - nn$$

$$s = mm + mn - nn; r = mm - 2mn = m(m - 2n)$$

vnde fit

$$p + s = 3mm - mn = m(3m - n)$$

$$p - s = mm - 3mn + 2nn = (m - n)(m - 2n)$$

$$q + r = 2mm - mn - nn = (2m + n)(m - n)$$

$$q - r = 3mn - nn = n(3m - n)$$

quibus valoribus in sequenti calculo vtemur.

13. His constitutis valoribus, pro numeratore nostrae fractionis habebimus:

$$pp + ss = 5m^4 - 6m^3n + 7m^2nn - 6mn^3 + 2n^4, \text{ seu}$$

$$pp + ss = (mm + nn)(5mm - 6mn + 2nn) \text{ et}$$

$$qq + rr = 2m^4 - 2m^3n + 3m^2nn - 2mn^3 + n^4, \text{ siue}$$

$$qq + rr = (mm + nn)(2mm - 2mn + nn)$$

vnde fractio nostra ad quadratum reducenda erit:

$$\frac{M}{N} = \frac{(5mm - 6mn + 2nn)(mm + nn)^2}{2n(2m + n) \cdot m^2 (m - n)^2 (m - 2n)^2 (3m - n)^2 (mm + mn - nn)^2}$$

hincque colligimus:

$$\frac{M}{N} = \frac{mm + nn}{m(m - n)(m - 2n)(3m - n)(mm + mn - nn)} \cdot \sqrt{\frac{5mm - 6mn + 2nn}{2n(2m + n)}}$$

to-

totum ergo negotium huc est reductum, vt formula
 $\frac{5mn - 6mn + 2nn}{2n(2m + n)}$ quadratum efficiatur, id quod infinitis modis
 praestari posse manifestum est, statim atque vnicus casus
 innotuerit.

14. Quo haec forma tractabilior reddatur, ponamus
 $2m - n = l$, vt sit $n = 2m - l$ et formula ad quadra-
 tum reducenda erit: $\frac{mm - 2ml + 2ll}{(4m - 2l)(4m - l)}$, vbi productum ex nu-
 meratore in denominatorem euolutum quippe quod etiam
 quadratum esse debet, perducit ad hanc conditionem

$$16m^4 - 44m^3l + 58mml - 28ml^3 + 4l^4 = \square$$

cuius quum ambo termini extremi iam sint quadrati per
 methodos satis cognitae facile est innumerabiles solutiones
 inuestigare; quem in finem ponamus $\frac{m}{l} = z$ vt habeamus
 hanc formulam $16z^4 - 44z^3 + 58zz - 28z + 4 = \square$;
 quae ponendo $z = y - 2$; transit in hanc:

$$16y^4 - 172y^3 + 706yy - 1300y + 900 = \square$$

vbi iterum ambo extremi termini sunt quadrata.

15. Ad hoc negotium expediendum, praestabit resolu-
 tionem nostrae aequationis siue prioris, siue posterioris in ge-
 nere docere. Sit igitur proposita haec aequatio generalis:

$$a\alpha z^4 - 2\beta z^3 + \gamma zz - 2\delta z + \varepsilon\varepsilon = \square;$$

atque pro idoneis valoribus ipsius z sequentes quatuor
 formulae per methodos consuetas reperiuntur:

$$\begin{aligned}
\text{I. } z &= \frac{2\alpha(\beta\epsilon - \alpha\delta)}{2\alpha^3\epsilon + \beta\beta - \alpha\alpha\gamma} \\
\text{II. } z &= \frac{2\alpha\epsilon^3 + \delta\delta - \gamma\epsilon\epsilon}{2\alpha\epsilon^3 + \delta\delta - \gamma\epsilon\epsilon} \\
\text{III. } z &= \frac{2\epsilon(\alpha\delta - \beta\epsilon)}{(2\alpha^3\epsilon + \alpha\alpha\gamma - \beta\beta)(2\alpha^3\epsilon - \alpha\alpha\gamma + \beta\beta)} \\
\text{IV. } z &= \frac{4\alpha\alpha(2\alpha^4\delta - \alpha\alpha\beta\gamma + \beta^3)}{4\epsilon\epsilon(2\beta\epsilon^4 - \gamma\delta\epsilon\epsilon + \delta^3)} \\
&= \frac{(2\alpha\epsilon^3 + \gamma\epsilon\epsilon - \delta\delta)(2\alpha\epsilon^3 - \gamma\epsilon\epsilon + \delta\delta)}{(2\alpha\epsilon^3 + \gamma\epsilon\epsilon - \delta\delta)(2\alpha\epsilon^3 - \gamma\epsilon\epsilon + \delta\delta)}
\end{aligned}$$

vbi quum litterae α et ϵ pro lubitu tam positue quam negatiue accipi queant, binæ priores formulae geminos valores suppeditant.

16. Quemadmodum autem innumerabiles huius aequationis solutiones inueniri oporteat, sequenti modo calculus instituat. Sit f valor quicunque per praecedentes formulas inuentus, ita vt nostra expressio

$$\alpha\alpha z^4 - 2\beta z^3 + \gamma z z - 2\delta z + \epsilon\epsilon,$$

posito $z = f$ fiat quadratum, sitque propterea

$$\alpha\alpha f^4 - 2\beta f^3 + \gamma f f - 2\delta f + \epsilon\epsilon = g g;$$

nunc igitur ponatur $z = x + f$ et nostra aequatio induet hanc formam:

$$\begin{aligned}
&\alpha\alpha x^4 + 4\alpha\alpha x^3 + 6\alpha\alpha f f + 4\alpha\alpha f^3 + g g = \square \\
&\quad - 2\beta \quad - 6\beta f x x - 6\beta f f x \\
&\quad \quad + \gamma \quad \quad + 2\gamma f \\
&\quad \quad \quad - 2\delta
\end{aligned}$$

quae aequatio breuitatis gratia ita repraesentetur:

$$\alpha\alpha x^4 - 2bx^3 + cxx - 2dx + ee = \square$$

ita vt sit $\alpha\alpha = \alpha\alpha$; $b = \beta - 2\alpha\alpha$; $c = \gamma - 6\beta f + 6\alpha\alpha f f$,
 $d = \delta - \gamma f + 3\beta f f - 2\alpha\alpha f^3$; ac denique $ee = gg$, vbi

sumi

sumi potest $a = \pm \alpha$ et $e = \pm g$. Tum vero quatuor
noui valores pro x inueniuntur sequentes:

$$\text{I. } x = f + \frac{2a(5e - ad)}{2a^3e + 5b^2 - aac}$$

$$\text{II. } x = f + \frac{2ae^3 + dd' - ce'e}{2e(2a - be)}$$

$$\text{III. } x = f + \frac{(2a^3e + aac - 5b)(2a^3e - aac + 5b)}{4aa(2a^4d - aabc + b^3)}$$

$$\text{IV. } x = f + \frac{4e'e(2be^4 - cde'e + d^3)}{(2ae^3 + ce'e - dd)(2ae^3 - ce'e - dd)}$$

quoniam igitur quemcunque valorem pro x hoc modo in-
uentum assumere licet, hinc numerus solutionum in infini-
tum augeri poterit.

17. Postquam autem pro x valor quicunque idoneus
fuerit inuentus, qui sit $x = \frac{b}{k}$, ob $x = \frac{m}{l} = \frac{m}{2m-n}$, habe-
bimus $m = k$ et $n = 2k - k$, ex quibus duobus numeris m
et n reliquae quantitates sequenti modo determinantur:

$$p = 2mm - 2mn + nn; \quad q = mm + mn - nn;$$

$$s = mm + mn - nn; \quad r = mm - 2mn = m(m - 2n),$$

ubi notasse iurabit esse:

$$pp + ss = (mm + nn)(5mm - 6mn + 2nn) \quad \text{et}$$

$$qq + rr = (mm + nn)(2mm - 2mn + nn) = (mm + nn)p,$$

atque hinc denique ambo nostri numeri quaesiti erunt

$$A = \frac{(mm + nn)^2(5mm - 6mn + 2nn)}{4m(m - 2n)(mm + mn - nn)^2} \quad \text{et}$$

$$B = \frac{(mm + nn)^2(5mm - 6mn + 2nn)(2mm - 2mn + nn)}{(3m - n)^2(m - n)^2mn(m - 2n)(2m + n)}.$$

18. Ut autem etiam innotescat, quemadmodum huius-
modi valores inuenti satisfaciant, ex binis numeris idoneis

mi

m et n prodeat formula radicalis $\sqrt{\frac{5mm-6mn+2nn}{2n(2m+n)}} = \frac{\mu}{v}$, vnde colligitur $\frac{M}{N} = \frac{(mm+mn) \mu}{v m(m-n)(m-2n)(3m-n)(mm+mn-nn)}$, tum vero quoniam supra litteras q et r permutauimus, quaternae formulae propositae, sequenti modo ad quadrata reducentur

$$\begin{aligned} \text{I. } \sqrt{(AB+A+B)} &= \frac{M}{N}(pr+qs) = \frac{\mu}{v} \cdot \frac{(mm+nn)^2}{m(m-2n)(mm+mn-nn)} \\ \text{II. } \sqrt{(AB+A-B)} &= \frac{M}{N}(pr-qs) = \frac{\mu}{v} \cdot \frac{(mm+nn)(m^4-8m^3n+6mmnn-n^4)}{m(m-n)(m-2n)(3-mn)(mm+mn-nn)} \\ \text{III. } \sqrt{(AB-A+B)} &= \frac{M}{N}(pq+rs) = \frac{\mu}{v} \cdot \frac{(m-n)(m-2n)}{(mm+nn)} \\ \text{IV. } \sqrt{(AB-A-B)} &= \frac{M}{N}(pq-rs) = \frac{\mu}{v} \cdot \frac{(mm+nn)}{m(3m-n)}. \end{aligned}$$

Aliae transformationes formulae resoluendae.

19. Quum tota quaestio huc sit perducta, vt ista formula (13) $\frac{5mm-6mn+2nn}{2n(2m+n)}$, siue $\frac{(2m-n)^2 + (m-n)^2}{2n(2m+n)}$ ad quadratum reuocetur, ponamus $2m-n=t$ et $m-n=u$, ita vt sit $m=t-u$ et $n=t-2u$, hincque $2m+n=3t-4u$ atque nunc quadratum esse debeat $\frac{tt+uu}{(2t-4u)(3t-4u)} = \square$, siue $\frac{tt+uu}{(4u-2t)(4u-3t)} = \square$ circa quam formulam obseruo, numeratorem cum denominatore alios factores communes habere non posse praeter 2 et 5. Hinc igitur sequitur numeratorem $tt+uu$ vel ipsum quadratum esse debere vel duplum, vel quintuplum vel decuplum quadratum. Vnde quatuor casus resultant, quos singulos sequenti modo euoluamus.

20. Denotent litterae a et b binos cathetos trianguli rectanguli numerici, cuius hypotenusa sit $=c$, ita vt sit $aa+bb=cc$, nunc igitur pro primo casu faciamus

$tt+$

$tt + uu = cc$, quod fit sumendo $t = a$ et $u = b$, atque hoc casu necesse est, vt fiat $(4b - 2a)(4b - 3a) = \square$.

Pro II^{do} Casu faciamus $tt + uu = 2cc$, quod fit sumendo $t = a - b$ et $u = a + b$, atque nunc necesse est vt sit $(a + 3b)(a + 7b) = \square$.

Pro III^{io} Casu faciamus $tt + uu = 5cc$, quod fit sumendo $t = a + 2b$ et $u = 2a - b$; tum enim ob $4u - 2t = 4a - 8b$ et $4u - 3t = 5a - 10b$, formula ad quadratum reducenda erit $(6a - 8b)(a - 2b) = \square$, hoc est $(4b - 2a)(4b - 3a) = \square$, quae cum Casu I^{mo} perfecte congruit.

Pro Casu denique IV^{to}, faciamus $tt + uu = 10.cc$, quod fit sumendo $t = 3a + b$ et $u = a - 3b$, tum enim ob $4u - 2t = -14b - 2a$, et $4u - 3t = -5a - 15b$, formula ad quadratum reducenda erit $(3b + a)(7b + a) = \square$, prorsus vti in casu secundo. Verum hic notandum est, casum tertium et quartum adhuc alio modo expediri posse. Si enim pro tertio ponamus $t = a + 2b$ et $u = b - 2a$, ob $4u - 2t = -10a$ et $4u - 3t = -2b - 11a$ formula ad quadratum reducenda erit $2a(11a + 2b) = \square$.

Pro Casu quarto autem, si ponamus $t = 3a + b$ et $u = 3b - a$, ob $4u - 2t = 10b - 10a$ et $4u - 3t = 9b - 13a$, formula ad quadratum reducenda est $(a - b)(13a - 9b) = \square$. Verum plerumque quoties his duobus casibus satisfieri pot-

est toties numeri t et u communi factore 5 praediti reperiuntur, ideoque ad nouas solutiones non perducunt.

21. His igitur duobus casibus postremis relictis, circa quatuor praecedentes omnino memoratu dignum est, quod primus et tertius, tum vero etiam secundus et quartus ad eandem formulam perduxerit, quare pro primo et tertio, si numeri a et b ita fuerint comparati, vt formula $(4b - 2a)(4b - 3a)$ fiat quadratum, tum duplici modo inde idonei valores pro t et u obtinentur; priori enim modo habebimus $t = a$ et $u = b$, altero vero modo $t = a + 2b$ et $u = 2a - b$. Simili modo pro casibus secundo et quarto, si fuerit formula $(3b + a)(7b + a)$ quadratum, tum etiam duo casus oriuntur, alter $t = a - b$ et $u = a + b$, alter vero $t = 3a + b$ et $u = a - 3b$. Operae igitur pretium erit has geminas resolutiones accuratius exponere.

I. Si fuerit $(4b - 2a)(4b - 3a) = \square$, existente $aa + bb = cc$.

22. Hinc igitur primo statim deducimus fractionem supra (18) introductam $\frac{\mu}{v} = \frac{cc}{(4b - 2a)(4b - 3a)}$; deinde pro priori resolutione habebimus

$$t = a; m = a - b$$

$$u = b; n = a - 2b$$

$$p = aa - 2ab + 2bb; r = (a - b)(3b - a)$$

$$q = aa - ab - bb; s = aa - ab - bb$$

$$\frac{p}{s} =$$

$$\frac{p}{s} = \frac{aa - 2ab + 2bb}{aa - ab - bb}; \quad \frac{q}{r} = \frac{aa - ab - bb}{(a-b)(3b-a)}$$

$$mm + nn = 2aa - 6ab - 5bb$$

pro altera vero solutione

$$t = a + 2b; \quad m = 3b - a;$$

$$u = 2a - b; \quad n = 4b - 3a;$$

$$p = 5(aa - 2ab + 2bb); \quad r = -5(a - b)3b - a)$$

$$q = -5(aa - ab - bb); \quad s = -5(aa - ab - bb)$$

$$\frac{p}{s} = \frac{aa - 2ab + 2bb}{aa - ab - bb}; \quad \frac{q}{r} = \frac{aa - ab - bb}{(a-b)(3b-a)}$$

unde manifestum est has duas solutiones a se inuicem non differre.

23. Speciales autem solutiones. quae ex hac formula primo intuitu deriuantur sunt sequentes

a	b	m	n	$\frac{p}{s}$	$\frac{q}{r}$
0	1	-1	-2	$\frac{2}{1}$	$\frac{1}{3}$
4	3	1	-2	$\frac{2}{1}$	$\frac{2}{7}$
12	5	7	2	$\frac{2}{1}$	$\frac{59}{21}$

quarum binae priores scopo nostro non conueniunt, tertia vero idoneam praebet solutionem atque adeo ab illa, quam olim iam inueni diuersam; quum enim sit $pp + ss = 8957 = 53 \cdot 169$ et $qq + rr = 3922 = 53 \cdot 74$ erunt ambo quaesiti numeri

$$A = \frac{160 \cdot 53^2 \cdot 74}{4 \cdot 74 \cdot 59^2 \cdot 21} = \frac{160 \cdot 53^2}{4 \cdot 21 \cdot 59^2}$$

$$B = \frac{169 \cdot 74 \cdot 53^2}{2 \cdot 16 \cdot 3 \cdot 5^2 \cdot 7 \cdot 19^2} = \frac{169 \cdot 37 \cdot 53^2}{16 \cdot 3 \cdot 5^2 \cdot 7 \cdot 19^2}$$

24. Consideremus autem attentius hanc formulam:
 $(4b - 2a)(4b - 3a) = \square$ et quia numeri a et b , sunt ca-

F 2

theti

theti trianguli rectanguli, atque euidens est, pro a sumi debere parem pro b vero imparem, statuamus $a = 2de$ et $b = dd - ee$, vt sit hypotenusa $c = dd + ee$, tum vero erit $4b - 2a = 4(dd - de - ee)$ et $4b - 3a = 4dd - 6de - 4ee$, quorum productum quum quadratum esse debeat, necesse est, vt vtriusque quadrans fiat quadratum, hoc est

$$\text{I}^{\circ}. dd - de - ee = \square$$

$$\text{II}^{\circ}. dd - \frac{3}{2}de - ee = \square,$$

vbi quum numerorum d et e alter debeat esse par, alter impar, etiam posterior numeris integris constat. Quod autem ad priorem attinet, quum sit $dd - de - ee = (d - \frac{1}{2}e)^2 - 5\frac{e^2}{4}$, ponamus $d - \frac{1}{2}e = rr + 5ss$ et $\frac{1}{2}e = 2rs$, tum enim fiet $dd - de - ee = (rr - 5ss)^2$; at vero habebimus $e = 4rs$ et $d = rr + 2rs + 5ss$ hincque $dd - ee = r^4 + 4r^3s - 2rrss + 20rs^3 + 25s^4$ et $de = 4r^3s + 8rrss + 20rs^3$, vnde altera conditio postulat: $r^4 - 2r^3s - 14rrss - 10rs^3 + 25s^4 = \square$.

25. Statuamus hic $\frac{r}{s} = z$, vt habeamus hanc formulam $z^4 - 2z^3 - 14zz - 10z + 25 = \square$, quae cum formula supra data (15) comparata praebet: $\alpha = \pm 1$; $\beta = 1$; $\gamma = -14$; $\delta = 5$; $\epsilon = \pm 5$, vnde pro z quatuor sequentes expressiones

$$\text{I}^{\circ}. z = \frac{2\alpha(\epsilon - 5\alpha)}{2\alpha^3\epsilon + 1 + 14} = \frac{2\alpha(\epsilon - 5\alpha)}{2\alpha^3\epsilon + 15} = \frac{2(\alpha\epsilon - 5)}{2\alpha\epsilon + 15}$$

hinc vel $z = 0$; vel $z = -4$

$$\text{II}^{\circ}. z = \frac{50 \cdot \alpha\epsilon + 375}{2(5\alpha\epsilon - 25)} = \frac{10 \cdot \alpha\epsilon + 75}{2(\alpha\epsilon - 5)} \text{ hincque}$$

vel

vel $z = \infty$; vel $z = -\frac{5}{4}$

$$\text{III}^{\circ}. \quad z = \frac{(2\alpha\epsilon - 14 - 1)(2\alpha\epsilon + 14 + 1)}{4(10 + 14 + 1)} = -\frac{125}{100} = -\frac{5}{4}.$$

$$\text{IV}^{\circ}. \quad z = \frac{100(1250 + 70 \cdot 25 + 5 \cdot 25)}{(50 \cdot \alpha\epsilon - 15 \cdot 25)(50 \cdot \alpha\epsilon + 15 \cdot 25)} = \frac{4 \cdot 25^2 \cdot 125}{25^2(2\alpha\epsilon + 15)(2\alpha\epsilon - 15)} = -4.$$

Ex valore $z = -4$ oriuntur valores $r = 4$; $s = -1$; $d = 13$; $e = -16$ hincque $a = 416$ et $b = 87$, vnde oritur $\frac{p}{s} = \frac{23362}{25859}$, et $\frac{q}{r} = \frac{25859}{10199}$; at ex valore $z = -\frac{5}{4}$, habemus $r = 5$; $s = -4$; $d = 65$; $e = -80$, qui per quinarium ad terminos minores reducti praebent vt ante, $d = 13$ et $e = -16$, vbi notasse iuuabit ex his valoribus a et b praegrandes numeros pro p , q , r , s esse prodituros.

26. At circa binas illas formulas notasse iuuabit, vtramque etiam quadrato negativo aequari posse, verum tum solutio eadem exsurgit, nisi quod valores pro a et b fiant negatiui. Ceterum hic notari conuenit, vltimae aequationi etiam valorem $z = -3$ satisfacere; etiamsi eum non per methodum consuetam detexerimus, inde autem fit $r = 3$ et $s = -1$; hincque porro $d = 2$ et $e = -3$; vnde fiat $a = -12$ et $b = -5$, quem casum iam supra euoluimus.

II. Si fuerit $(3b + a)(7b + a) = \square$.

27. Hic statim apparet sumi debere $a = dd - ee$ et $b = 2de$, vt fiat $c = dd + ee$ tum ergo sequentes duae formulae quadrata esse debent $dd + 6de - ee = \square$ et $dd + 14de - ee = \square$.

Quum

Quum prior sit $= (d+3e)^2 - 10ee$; si ponamus $\zeta\eta = 10$, ac statuamus $d+3e = \zeta rr + \eta ss$ et $e = 2rs$ fiet illa formula $= (\zeta\zeta rr - \eta\eta ss)^2$, tum autem erit $d = \zeta rr - 6rs + \eta ss$ et $e = 2rs$; hinc ergo pro altera formula, quae est $(d+7e)^2 - 50.ee$, erit $d+7e = \zeta rr + 8rs + \eta ss$ ideoque haec formula abit in $\zeta\zeta r^4 + 16\zeta r^3s - 116rrss + 16\eta rs^3 + \eta\eta s^4 = \square$, vnde per methodum supra indicatam infinitae solutiones inueniri possunt; vbi notasse iuuabit esse vel $\zeta = 1$ et $\eta = 10$, vel $\zeta = 2$ et $\eta = 5$.

28. Quum autem idonei valores pro a et b fuerint inuenti, duplici modo inde litterae t et u definiri poterunt. Priore modo fit $t = a - b$ et $u = a + b$, hinc $m = t - u = -2b$ et $n = -a - 3b$, ideoque $p = mm + (m-n)^2 = aa + 2ab + 5bb$; $q = s = mm + n(m-n) = -aa - 4ab + bb$ et $r = m(m-2n) = -4b(a+2b)$ ita vt sit

$$\frac{p}{s} = \frac{aa+2ab+5bb}{aa+4ab-bb}; \text{ et } \frac{q}{r} = \frac{aa+4ab-bb}{4b(a+2b)}.$$

Posteriore vero modo fit $t = 3a + b$ et $u = a - 3b$, vnde $m = 2a + 4b$ et $n = a + 7b$, hincque porro ob $m-n = a-3b$, fit $p = 5(aa + 2ab + 5bb)$ $q = s = 5(aa + 4ab - bb)$ et $r = 5 \cdot 4b(a+2b)$ sicque patet hunc posteriorem casum ad priorem redire.

29. Simpliciores autem solutiones, quas facili negotio diuinando elicere licet sunt sequentes:

a	b	m	n	$\frac{p}{s}$	$\frac{q}{r}$
1	0	— 0	— 1	$\frac{1}{1}$	$\frac{1}{0}$
— 3	4	— 8	— 9	$\frac{18}{11}$	$\frac{11}{10}$
— 35	12	— 24	— 1	$\frac{106}{139}$	$\frac{699}{628}$

Hic secundus casus praebeat illam ipsam solutionem, quam iam olim dederam. His autem duabus formulis pertractatis adiungamus insuper binas postremas supra (20) inuentas.

III. Si fuerit $2a(11a + 2b) = \square$.

30. Casus simpliciores, qui statim se offerunt sunt:

a	b	m	n	$\frac{p}{s}$	$\frac{q}{r}$
0	1	1, 1	0, 0	$\frac{2}{1}$	$\frac{1}{1}$
4	3	15, 3	20, 4	$\frac{2}{1}$	$\frac{1}{3}$
16	— 63	— 15, — 3	80, 16	$\frac{2}{1}$	$\frac{49}{11}$

vbi ex datis a et b , fit $t = a + 2b$ et $u = b - 2a$ hincque, vt ante $m = t - u = 3a + b$ et $n = t - 2u = 5a$. Hae solutiones autem iam in superioribus continentur.

IV. Si fuerit $(a - b)(13a - 9b) = \square$.

31. Inuentis idoneis valoribus pro a et b , erit $t = 3a + b$ et $u = 3b - a$, hinc $m = 4a - 2b = 2(2a - b)$ et $n = 5(a - b)$, atque ob $m - n = 3b - a$, atque $m - 2n = 2(4b - 3a)$ habebimus $\frac{p}{s} = \frac{17aa - 22ab + 13bb}{11aa + 4ab - 11bb}$ et $\frac{q}{r} = \frac{11aa + 4ab - 11bb}{4(6aa - 11ab + 4bb)}$. Solutiones autem simpliciores hinc oriundae sunt

a	b	m	n	$\frac{p}{s}$	$\frac{q}{r}$
0	1	— 2	— 5	$\frac{11}{11}$	$\frac{11}{16}$
4	+ 3	10, 2	5, 1	$\frac{1}{1}$	$\frac{1}{6}$

vbi memoratu dignum euenit, quod statim primum tentamen

men quo $a = 0$ et $b = 1$, praebeat solutionem iam dudum inuentam.

32. Quod si pro vltiore huius formulae euolutione ponamus $a = 2de$ et $b = dd - ee$, fiet $a - b = ee + 2de - dd$ siue mutandis signis, vt $(b - a)(9b - 13a) = \square$, erit $b - a = dd - 2de - ee$ et $9b - 13a = 9dd - 26.de - 9ee$, reddamus nunc priorem quadratum, quae quum sit $(d - e)^2 - 2ee$, statuamus $d - e = rr + 2ss$ et $e = 2rs$, tum enim fiet $dd - 2de - ee = (rr - 2ss)^2$, tum vero alter factor ob $dd - ee = r^4 + 4r^3s + 8rrss + 8rs^3 + 4s^4$, erit $9r^4 - 16r^3s - 68rrss - 32rs^3 + 36.s^4$, vbi casus primo intuitu se offerentes sunt 1°. $r = 1$, $s = 0$, 2°. $r = 0$, $s = 1$, 3°. $r = 1$ et $s = -1$, 4°. $r = 2$ et $s = -1$; 5°. $r = 1$ et $s = 2$.

33. Pro horum casuum primo habemus $d = 1$ et $e = 0$; hinc $a = 0$ et $b = 1$, qui iam occurrit, pro secundo habemus $d = 2$ et $e = 0$, hinc $a = 0$ et $b = 1$, qui a praecedente non differt. At pro tertio habemus $d = 1$ et $e = -2$, hinc $a = -4$ et $b = -3$, qui supra iam est tractatus, pro quarto habemus $d = 2$ et $e = -4$ siue $d = 1$ et $e = -2$, vnde fit $a = -4$ et $b = -3$ vt praecedens, pro quinto denique habemus $d = 13$ et $e = 4$, hinc $a = 104$ et $b = 153$, ex quibus numeri praegrandes pro quaesitis A et B resultant, quibus non immoramur.

34. Imprimis autem quoque notatu dignus est casus, quo inuenimus $\frac{p}{s} = \frac{2}{1}$ et $\frac{q}{r} = \frac{1}{3}$, siue $\frac{q}{r} = \frac{3}{1}$, vnde deducuntur numeri quaesiti $A = \frac{25}{12}$ et $B = \frac{25}{12}$ ita vt ambo numeri quaesiti hoc casu fiant aequales, quod quidem scopo problematis minus conuenit. Si enim numeri aequales desiderentur ob eorum differentiam euanescentem quaestio huc rediret, vt inueniatur numerus A , ita vt tam $AA + 2A$, quam $AA - 2A$ fiat quadratum, quod quidem est facillimum, statuatur enim $AA = \frac{aa+bb}{nn}$ et $2A = \frac{2ab}{nn}$, fiet vti que $\sqrt{(AA + 2A)} = \frac{a+b}{n}$ et $\sqrt{(AA - 2A)} = \frac{a-b}{n}$; verum nunc requiritur vt $aa + bb$ sit quadratum, quem in finem ponamus, $a = pp - qq$ et $b = 2pq$, vt fiat $A = \frac{pp+qq}{n}$, est vero etiam $A = \frac{2pq(pp-qq)}{nn}$, vnde fit $n(pp+qq) = 2pq(pp-qq)$ et $n = \frac{2pq(pp-qq)}{pp+qq}$, ita vt numerus quaesitus in genere sit $A = \frac{(pp+qq)^2}{2pq(pp-qq)}$, tales ergo numeri sunt sequentes: $A = \frac{25}{12}$; 2°. $A = \frac{169}{60}$; 3°. $A = \frac{289}{120}$; 4°. $A = \frac{625}{168}$ etc.

35. Pro solutionibus autem ad quaestionem propositam accommodatis, duae in numeris non nimis magnis notatu dignae videntur, quarum prior est ea ipsa, quam iam dudum inueni, qua erat $A = \frac{13 \cdot 29^2}{8 \cdot 9^2}$ et $B = \frac{5 \cdot 29^2}{32 \cdot 11^2}$, siue $A = \frac{10923}{648}$ et $B = \frac{4205}{3872}$ vnde $\sqrt{(AB + A + B)} = \frac{7 \cdot 29 \cdot 37}{16 \cdot 9 \cdot 11}$

$$\sqrt{(AB + A + B)} = \frac{29^2}{16 \cdot 3 \cdot 11}$$

$$\sqrt{(AB - A + B)} = \frac{29^2}{16 \cdot 9}$$

$$\sqrt{(AB - A - B)} = \frac{29}{48}.$$

Pro altera vero solutione orta ex valoribus:

$$\frac{p}{s} = \frac{74}{59} \text{ et } \frac{q}{r} = \frac{59}{21} \text{ obtinemus:}$$

$$A = \frac{13^2 \cdot 53^2}{4 \cdot 21 \cdot 59^2} \text{ et } B = \frac{13^2 \cdot 37 \cdot 53^2}{16 \cdot 3 \cdot 5^2 \cdot 7 \cdot 19^2}$$

$$\text{vnde } \sqrt{AB + A + B} = \frac{13 \cdot 53}{8 \cdot 3 \cdot 7}$$

$$\sqrt{AB + A - B} = \frac{13 \cdot 53^2}{8 \cdot 3 \cdot 5 \cdot 7 \cdot 19}$$

$$\sqrt{AB - A + B} = \frac{13 \cdot 53^2}{8 \cdot 3 \cdot 7 \cdot 59}$$

$$\sqrt{AB - A - B} = \frac{13 \cdot 47 \cdot 47 \cdot 53}{8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 59}$$

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